

3. ALGEBRA

§3.1. Pronouns and Pronumerals

The language of mathematics is not completely separate from English, in the way that Japanese is. It is an extension of English, or Japanese, or any other natural language. These extensions allow us to express mathematical ideas with precision and to communicate mathematically with other people.

Like natural languages such as English, mathematics allows us to think. Without language we would not be able to comprehend abstract thoughts and our thinking would remain on the level of “I am hungry”, “I am afraid” or “I am cold”.

Many of the rules and conventions of a natural language apply to mathematics. Every sentence must have a verb. A fragment of language without a verb is a phrase. The most common verb in mathematics is “equals”, which we write as “ $=$ ”. So $2 + 2 = 4$ is a mathematical sentence. $2 + 2$ is simply a mathematical phrase, although in mathematics such fragments usually go under the name of **expression**.

Now in natural languages we have nouns and pronouns. Numbers are not the only nouns in mathematics, but they are the most common – especially at the elementary level. But in English we have pronouns such as ‘he’, ‘it’, ‘they’. All other natural languages have

their pronouns. The role of a pronoun is to stand for a noun.

When we say “he trashed the house” the word ‘he’ stands for a specific person, though we may not know who ‘he’ is. When we say “those who choose not to work are doomed to live an unfulfilling life” the pronoun ‘those’ refer to whoever fits the description.

Corresponding to pronouns in natural languages, in mathematics we have **pronumerals**. These are symbols that represent numbers (or in more advanced mathematics, other mathematical objects). We use symbols such as x , y or z . These symbols represent numbers, but unlike π that has a fixed meaning, a symbol such as x can represent any possible number.

We sometimes call x an **unknown**. Often we call it a **variable**. These two words represent the two reasons why we write x instead of a specific number, such as 42. It may be that we are making a general statement, like “those who choose to work are doomed to live an unfulfilling life”. Here we are thinking of the x and y as representing arbitrary numbers. A mathematical sentence of this type is $x + y = y + x$. We could express this in words by saying “when you add any two numbers you get the same answer if you swap them around”. The mathematical version is so much more compact, and moreover, it means the same to a Japanese mathematician as to a Swedish one. The language of mathematics is universal.

Of course, when we say “the language of mathematics” we are not being quite accurate. Mathematics uses natural language as well as specialised symbols, so it is an add-on to a natural language and so German mathematics books need to be translated into English and vice versa.

The other use for pronumerals is in solving equations. In an equation such as $3x + 2 = 14$ the x represents an unknown number. Solving an equation is a bit like a who-dun-it! There is a number such that if you double it and add 2 you get 14. What is the number?

§3.2. Variable Names

You’ll notice that in the above equation:

$$3x + 2 = 14$$

I wrote three times x as $3x$. The multiplication sign is not included. Apart from the fact that a multiplication sign could be mistaken for an x .

Imagine if we wrote $3 \times x + 2 = 4$. Besides looking strange, $3x$ is more compact than $3 \times x$. Well, why don’t we leave out the multiplication sign when multiplying two numbers? We can’t do that because writing 3 times 4 as 34 would make us think that it was 34 instead of 12.

There’s a special case when we have to multiply by 1. Instead of writing $1x$ (or worse still, $1 \times x$) we just write x . So $x + 5 = 6$ is saying “there’s a number which, when added to 5, becomes 6”. Squares and square roots are written the same with x ’s as with actual numbers. The

square of x , or x squared, is x times x and is written as x^2 and the square root of x is written as \sqrt{x} .

Often we use different letters of the alphabet, such as y and z . But keep in mind that even if we use a different symbol the value might be the same. For example, when we write the equation $x + y = xy$ we include the possibility that $x = y = 2$. [There are other solutions where x, y are different such as $x = 3, y = \frac{3}{2}$.]

But we don't have to stick to x, y and z for our variables and unknowns. Any letter of the alphabet will do, and we can even use capital letters in addition to lower case ones. So we might write the area of a circle as $A = \pi r^2$, where A represents the area and r represents the radius of the circle. Of course π is not a variable. It is the constant that we talked about earlier.

But mathematicians found the Roman alphabet $a - z$ too restrictive, even allowing for capital letters which might represent different numbers to their lower case counterparts. So they plundered the Greek alphabet for suitable variable names. Here are some of the Greek lower case symbols that are commonly used.

α	β	γ	δ	χ	ϕ
alpha	beta	gamma	delta	chi	phi

λ	μ	θ	ρ	σ	ω	ψ
lambda	mu	theta	rho	sigma	omega	psi

You'll notice that the Greek letter π is not used as a variable name. That's because it has a special meaning for a particular number 3.14159 ...

There are other Greek lower-case letters, but they are too similar to ordinary letters, so we don't use them. And, of course, we can use upper-case Greek letters. Here are some of the more commonly used ones.

Δ	Σ	Ω
delta	sigma	omega

But even that is sometimes not enough. We can combine a letter and a number to create a double-barrelled name. If we want 100 variable names we might choose x_1, x_2, \dots, x_{100} . Subscripts, unlike superscripts that represent powers, don't combine the variable with the number in some arithmetic combination. It is just like using a given name and a surname for a single individual.

But, having seen what a variety of variable names we have available, we will stick mainly to x and y for quite some time.

§3.3. Doing Arithmetic With Algebraic Expressions

An **algebraic expression** is like a phrase in the language of mathematics. Let us consider algebraic expressions in two variables such as $3x + 2y$. How do we carry out arithmetic with these: adding, subtracting,

multiplying and dividing? With addition and subtraction we treat the x and y as we would two different types of fruit.

If you have 3 apples and 2 oranges and I have 4 apples and 9 oranges, and we combine our collections, we would have 7 apples and 11 oranges. So we just add the numbers of apples and the numbers of oranges. We do the same with algebraic expressions.

Example 1: Add $3x + 2y$ to $4x + 9y$.

Solution: $(3x + 2y) + (4x + 9y) = 7x + 11y$.

The parentheses here are not necessary. We use them merely to show how the expressions came from two collections.

Example 2: Subtract $3x + 2y$ from $4x + 9y$.

Solution: $(4x + 9y) - (3x + 2y) = x + 7y$.

This is no different from asking what would I have left if I had 4 apples and 9 oranges and I gave you 3 apples and 2 oranges. Notice that the parentheses are necessary here. If we wrote it as $4x + 9y - 3x + 2y$ we would have $x + 11y$.

Multiplying expressions it is important that we use parentheses. We have to treat everything inside a pair of parentheses in the same way. Each term in one bracket has to multiply each term in the other.

Example 3: Multiply $3x + 2y$ by $4x + 9y$.

Solution: We must write this as $(3x + 2y)(4x + 9y)$.

This gives four terms. (A **term** is a piece that is added to other pieces.)

$$(3x + 2y)(4x + 9y) = 12x^2 + 27xy + 8yx + 18y^2.$$

For example $(3x)(9y) = 27xy$. We always bring the numbers to the front. We cannot simplify xy . It has to be left as xy .

But, for numbers, $yx = xy$ so we can combine the terms $27xy$ and $8yx$ to get $35xy$.

$$\begin{aligned} \text{So } (3x + 2y)(4x + 9y) &= 12x^2 + 27xy + 8yx + 18y^2 \\ &= 12x^2 + 35xy + 18y^2. \end{aligned}$$

Here's a picture that represents this calculation. The area of a rectangle is the product of the length and the breadth. So $(3x + 2y)(4x + 9y)$ represents the area of a rectangle whose length is $3x + 2y$ and whose breadth is $4x + 9y$, or vice versa. This can be broken up into 4 smaller rectangles.

	$4x$	$9y$
$3x$	$12x^2$	$27xy$
$2y$	$8yx$	$18y^2$

Example 4: Divide $4x + 9y$ by $3x + 2y$.

Solution: The answer is simply $\frac{4x + 9y}{3x + 2y}$. We can't simplify it.

Now why don't we use parentheses here like we did with the product? There would be nothing wrong with writing $\frac{(4x + 9y)}{(3x + 2y)}$ but the horizontal bar implicitly puts parentheses around the numerator and denominator, so there is no need to explicitly include them.

A similar situation exists with the square root symbol. If we write $\sqrt{3x + 2y}$ we read this as the square root of the package that consists of $3x$ added to $2y$. It is unnecessary to write $\sqrt{(3x + 2y)}$.

In general some form of bracketing is necessary if we want to treat an algebraic expression as a single package. As well as the implicit bracketing of the fraction bar and the square root sign (and cube roots etc) we have the parentheses. These parentheses can be nested, such as in $(2(x + y) + z)(x + 3(y + z))$.

Here the $x + y$ is packaged up into a single package, and then doubled. Then z is added and this whole thing becomes a single package. The second package is x added to a sub-package that consists of $y + z$ multiplied by 3. Then these two packages are multiplied together.

Sometimes, for clarity, we use different types of bracketing. As well as parentheses, or round brackets, we can use square brackets or curly brackets.

So we could write the above expressions as:

$$[2(x + y) + z]\{x + 3(y + z)\}$$

but it is a little extreme to use all three types of brackets in one expression. It looks better if we write this as:

$$[2(x + y) + z][x + 3(y + z)].$$

This way we are using each type of bracket for a different level of bracketing. However the original expression $(2(x + y) + z)(x + 3(y + z))$ is perfectly valid.

Example 5: Simplify $(2(x + y) + z)(x + 3(y + z))$.

Solution: The best methodology is to work from inside out.

$$(2(x + y) + z)(x + 3(y + z)) = (2x + 2y + z)(x + 3y + 3z).$$

Because the $x + y$ is bracketed both x and y must be doubled. Similarly $3(y + z) = 3y + 3z$.

Now, when we expand we will get 9 terms, because each of the three terms in the first bracket has to multiply each of the three terms in the second.

$$\begin{aligned}(2x + 2y + z)(x + 3y + 3z) \\ &= 2x^2 + 6xy + 6xz + 2yx + 6y^2 + 6yz + zx + 3zy + 3z^2 \\ &= 2x^2 + 6y^2 + 3z^2 + 8xy + 7xz + 9yz.\end{aligned}$$

It's a good idea to write product of variables in alphabetic order. This makes it easier to gather together terms. This is why we change $2yx$ to $2xy$.

	x	3y	3z
2x	$2x^2$	$6xy$	$6xz$
2y	$2yx$	$6y^2$	$6yz$
z	Zx	$3zy$	$3z^2$

Often we get higher powers, such as x^2 or x^3 in the original expressions. We just remember that x^2 means xx and x^3 means xxx , so $x^2x^3 = xxxxx = x^5$.

When we multiply powers of the same variable we add the powers.

Example 6: Simplify $(x^2 + 3x + 5)(x^3 + 2x)$.

Solution: Here there will be 6 terms as each of the three terms in the first factor is multiplied by each of the two terms in the second.

$$\begin{aligned} (x^2 + 3x + 5)(x^3 + 2x) &= x^5 + 2x^3 + 3x^4 + 6x^2 + 5x^3 + 10x \\ &= x^5 + 3x^4 + 7x^3 + 6x^2 + 10x. \end{aligned}$$

An expression of this type is called a **polynomial** in x and it's normal practice to arrange the terms in descending powers of x .

§3.4. The Laws of Algebra

There are certain properties that the real numbers possess which underlie basic algebra.

The Commutative Laws:

For all real numbers:

$$a + b = b + a \text{ and } ab = ba.$$

You have known for a long time that it doesn't matter in which order you add a list of numbers, and you can multiply numbers in any order. But be warned. This is not the usual state of affairs in mathematics, or even in life for that matter. The order in which you carry out certain operations is often highly critical. If a watchmaker has to reassemble a watch it *does* matter in which order he puts in the pieces. Carrying out the operations of putting on your socks and your shoes you get quite a different result if you reverse the usual order!

There are mathematical objects which can be multiplied where $ab \neq ba$. This leads to all sorts of complications. So remember, despite the heading for this section we are not really discussing the laws of algebra in general – only the laws of the algebra of numbers.

The Associative Laws:

For all real numbers:

$$(a + b) + c = a + (b + c) \text{ and } (ab)c = a(bc).$$

This means that when we add three or more **terms** (expressions or numbers that are added together), or multiply three or more **factors** (expressions or numbers that are multiplied together), it does not matter how we group them. As a result we usually write the above as $a + b + c$ and abc respectively.

Again these are not laws that operate in all algebraic situations, though they do hold in most – certainly all the ones you are likely to meet. Because of the associative laws for real numbers we write things like $3x$, meaning $x + x + x$ and $x^3 = xxx$. If addition or multiplication were not associative these would be ambiguous. Does x^4 mean $x((xx)x)$, or is it $(xx)(xx)$. Or maybe even $((xx)x)x$? Fortunately, because of the Associative Law for multiplication, it doesn't matter.

Identities:

There are two real numbers that behave specially when it comes to addition and multiplication. These are the numbers 0 and 1.

For all real numbers x :

$$\mathbf{0 + x = x + 0 = x \text{ and } \mathbf{1x = x1 = x.}}$$

These numbers are called the **additive identity** (the number 0) and the **multiplicative identity** (the number 1).

Inverses:

For all real numbers x there's a number, written $-x$ such that $x + (-x) = (-x) + x = \mathbf{0}$. The number $-x$ is called the **additive inverse** of x . When it comes to multiplication we have to make an exception. For all non-zero real numbers x there's a number, written $1/x$ or x^{-1} , such that

$$\mathbf{xx^{-1} = x^{-1}x = \mathbf{1.}}$$

It is called the **multiplicative inverse** or **reciprocal**.

Why the exception? Why can't 0 have a multiplicative inverse? Why don't we write $\frac{1}{0} = \infty$? Quite apart from the difficulty of finding a point on the real line to represent it, our whole system would implode if we allowed this. This is because we would then have $0\infty = 1$. But what's wrong with that? The answer is "nothing, if you want to abandon the distributive laws". We insist on having the distributive law, so 0 will just have to do without an inverse. Let's see what the distributive law is.

Distributive Law:

The distributive law ties the additive and multiplicative structures of the real numbers together.

For all real numbers $a(b + c) = ab + ac$.

Because multiplication is commutative we can also write $(b + c)a = ba + ca$. This is a fundamental property of the real numbers and we use it every time we expand an expression.

Example 7: Expand $(3x + 2y)(5x + y)$.

Solution:

$$\begin{aligned}(3x + 2y)(5x + y) &= (3x + 2y)(5x) + (3x + 2y)(y) \\ &= 15x^2 + 10xy + 3xy + 2y^2 \\ &= 15x^2 + 13xy + 2y^2.\end{aligned}$$

Here we are using more than the distributive law. When we multiplied $3x$ by $5x$ to get $15x^2$ we unconsciously made

use of the associative law and the commutative law for multiplication. And the fact that we wrote the answer as:

$$15x^2 + 13xy + 2y^2$$

and not $(15x^2 + 13xy) + 2y^2$ or $15x^2 + (13xy + 2y^2)$

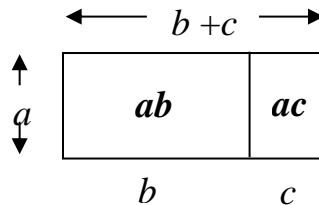
shows that we are mindful of the associative law for addition. Note too that in writing $10xy + 3xy$ as $13xy$ we are unconsciously making use of the distributive law. If we had to justify it we could write

$$10xy + 3xy = (10 + 3)xy = 13xy.$$

When we worked with algebraic expressions in high school we probably didn't think too much about what we were doing. We just instinctively changed one expression into an equivalent one. But, if we want to know *why* things work the way they do, we could justify everything from the above fundamental laws.

And what if we wanted a proof that these laws are correct? To do that we'd have to carefully define a real number. Even defining the number 2 in a formal way is quite a sophisticated matter. Now is not the time or place to get bogged down with such fundamentals.

However we can provide a geometric explanation of the distributive law. If you accept that ab is the area of an $a \times b$ rectangle then this picture demonstrates that $a(b + c) = ab + ac$.



Now there are many other laws of algebra that were not given in the above list. For example we all know that “if you multiply by zero, you get zero”. Is this yet one more law we have to accept? No, it is a simple consequence of the ones we’ve already given. However we won’t take time here to derive them. Two simple ones are $0x = 0$ and $(-1)(-1) = 1$.

It follows that the product of two negative numbers is positive, for example $(-3)(-2) = 6$. Perhaps your teacher might have explained it by saying that “two negatives make a positive” and referred to the fact that someone who is not unkind is kind. But that is simply a sleight of hand, because in other contexts two negatives do not make a positive – certainly not if you add two negative numbers. Still, in primary school a formal proof would be inappropriate.

The existence of an inverse under multiplication allows us to cancel by a non-zero number.

That is, if $xy = xz$ and $x \neq 0$ then $y = z$.

§3.5. Basic Algebraic Identities

There are certain equations that hold for all variables in the algebra of real numbers. Here are a couple that are useful to know.

Theorem 1: For all real numbers x, y :

(1) $(x + y)^2 = x^2 + 2xy + y^2$.

(2) $(x - y)(x + y) = x^2 - y^2$.

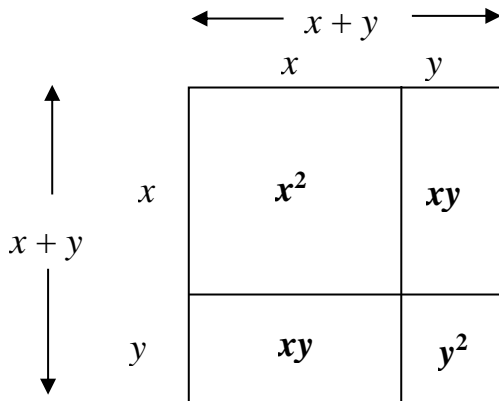
We can prove these, using the above laws of algebra, but you are probably happy to accept this fact on my authority.

The special cases of (2), where $y = 1$, should be noted as it arises frequently.

$$x^2 - 1 = (x - 1)(x + 1).$$

Also note that although $x^2 - 1$ factorises, $x^2 + 1$ does not.

The first of these identities can be illustrated geometrically as follows.



§3.6. Solving Equations

An equation is a mathematical sentence where the verb is the equals sign. It differs from an identity in that an identity holds for *all* values of the variables while an equation does not. If there is just one variable, such as x , we can solve the equation, which means find all the values for which the equation is true. Some equations have no solutions, others have just one solution. Yet other equations have several solutions and some even have infinitely many solutions.

A **linear equation** (in one variable) has the form:

$$ax + b = c.$$

If $a \neq 0$ we can solve for x , and there is just one solution.

Example 8: Solve $7x + 5 = 9$.

Solution: Subtract 5 from both sides, giving $7x = 4$. Divide both sides by 7, giving $x = 4/7$.

Often a problem is described in words and involves some real-world situation. In this case we must first express the problem as an equation, and then to solve it. In fact we can identify 6 separate steps.

(1) Define the variable or variables in the problem.

Don't start throwing x 's and y 's around without describing what quantities they represent.

(2) Clearly state the units you are using.

For example you might say "let x be the length in metres".

- (3) Summarise the facts by a diagram or a table.**
- (4) Write the facts as an equation, or set of equations.**
- (5) Solve.**
- (6) Interpret your solution in the context of the problem, using appropriate units and an appropriate level of approximation.**

Don't just finish with a statement like " $x = 3.5$ ". If a person came to you with the problem, and you decided to let x be the length in metres, the person asking the question wouldn't understand your saying " $x = 3.5$ ". You would need to say "the length is 3.5 metres".

If the answer was $x = 0.015$ you should not say "the length is 0.015 metres", but rather "the length is 1.5 cm". If the answer, using your calculator, came out as $x = 2.87451839201$, you shouldn't say "the length is 2.87451839201 metres", but rather "the length is 2.87 metres", or perhaps "the length is 2.874 metres" or perhaps, rounding up, "the length is 2.875 metres".

Not every mathematical solution to an equation translates to a solution to the original problem. Suppose that the equation had two solutions $x = 3$ and $x = -2$. If x was a length then clearly the solution $x = -2$ must be rejected.

Example 9: John is 5 years older than his sister. Fifteen years ago he was exactly twice his sister’s age. How old is John now?

Solution: Let x be John’s age now (we don’t need to say “in years” – that’s understood).

	John	sister
now	x	$x - 5$
then	$x - 15$	$x - 20$

Since John was exactly twice his sister’s age 15 years ago then:

$$\begin{aligned}
 x - 15 &= 2(x - 20) \\
 &= 2x - 40.
 \end{aligned}$$

$$\begin{aligned}
 \therefore 2x - x &= 40 - 15, \text{ which gives} \\
 x &= 25.
 \end{aligned}$$

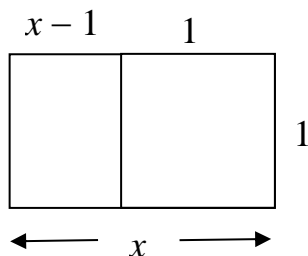
So John’s age now is 25.

This very simple problem could have been solved just as easily with a little bit of trial and error. But rarely is this the case.

Example 10: A 1 metre by 1 metre square is cut off the end of a rectangle. The resulting rectangle, when rotated 90° , has the same proportions as the original rectangle. What is the area of the original rectangle?

Solution: The width, or shorter side of the rectangle will be 1 metre.

Let x be the length of the longer side, in metres.



The ratio of the longer side to the shorter is $\frac{x}{1}$ for the original rectangle, and $\frac{1}{x-1}$ for the remaining rectangle.

Hence $\frac{x}{1} = \frac{1}{x-1}$.

Simplifying, this gives $x(x-1) = 1$, or $x^2 - x - 1 = 0$.

This is what is called a quadratic equation, and we will show how to solve these in a later chapter. Let me simply report that there are two solutions: $x = \frac{1 \pm \sqrt{5}}{2}$.

Using a calculator, these are $x = 1.618033989$ and $x = -0.618033989$. Clearly we must reject the negative solution. Moreover, for most purposes, the degree of accuracy given is far more than is appropriate. We should report that the longer side is about 1.62 metres, or perhaps 162 cm.

But the question doesn't ask for the length of the longer side, but the area. Since the shorter side is 1, the answer is still 1.62, but rather than metres, it is now *square* metres. When you think that you have finished always check your answer by asking three questions.

- (1) Is it a sensible solution?
- (2) Is it intelligible to someone who has read the question but none of the solution.
- (3) Have you answered what was asked for?

